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Archimedean Copulas Family via Hyperbolic Generator

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ABSTRACT

In this study the main endeavor is to generate an Archimedean copulas (AC) family via hyperbolic function. With using $csch$ function, a new generator of Archimedean family will be defined. Scatterplots, contour diagrams and also surface of the generated new Archimedean family will be shown for several values of its parameter.

Keywords: Archimedean Copulas, Generator, Hyperbolic functions, Kendall's tau.

1. INTRODUCTION

Sklar (1959) for the first time used the word copula, as mathematically sense, a function which allows us to combine univariate distributions to obtain a joint distribution with a particular dependence structure.

Copulas can be divided to the elliptical and Archimedean families [2]. Elliptical copulas have elliptical distributions, on the other hand, have an elliptical form and therefore are symmetric in the tails. Well known copulas in this family are the Gaussian and the student's copula. The Gaussian copula often used because has simpler form than student's copula. AC are very easy to construct, many parametric families belong to this class and have a great variety of different dependency structures, so many numbers of authors are working with these copulas and also most of them investigate the bivariate case.

Copulas widely used in several branches of science. For example, McNeil et al [3] Clemen and Reilly [4] and some other people used copulas in finance, Najjari and Unsal [5] applied copulas in meteorological data, Celebioglu [6] used copulas in modeling of students grades. Al-Harthy et al [7] used copulas to model dependence in petroleum decision making. Many other authors worked on copulas like, Genest and MacKay [8,9] Hua and Joe [10] among them.

In this study we generate a new family of AC via hyperbolic function. With using $csch$ ($cosech$) function, we define a convex and strictly decreasing function on $(0,1)$ then exhibit it can be a generator for an Archimedean family. Scatterplots, contour diagrams, and also surface of this new family will be shown for several values of its parameter.

This paper is constructed as follows: Section 2 discusses some about copulas and AC. Section 3, introduces new

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family of AC and investigates details of this new family. Section 4 summarizes the conclusion of our work.

2. PRELIMINARIES

We begin with the definition of bivariate copula. A copula is a function $C: [0,1]^2 \rightarrow [0,1]$ which satisfies:

(a) for every u, v in $[0,1]$, $C(u,0) = 0 = C(0,v)$ and $C(u,1) = u$ and $C(1,v) = v$.
(b) for every u_1, u_2, v_1, v_2 in $[0,1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0.$$

The importance of copulas in statistics is described in Sklar's theorem [1]: Let X and Y be random variables with joint distribution function H and marginal distribution functions F and G, respectively. Then there exists a copula C such that $H(x,y) = C(F(x), G(y))$ for all x, y in $\mathbb{R}$. Conversely if C is a copula and F and G are distribution functions, then the function H is a joint distribution function with margins F and G. If F and G are continuous then C is unique. Otherwise, the copula C is uniquely determined on $Ran(F) \times Ran(G)$. Conversely if C is a copula and F, G are distribution functions then the function H is a joint distribution function with margins F and G. As a result of the Sklar's theorem, copulas link joint distribution functions to their one-dimensional margins.

Let C be a copula and let u, v be any number in I then

$$W(u,v) \leq C(u,v) \leq M(u,v)$$

where

$$W(u,v) = max(u + v - 1, 0), \quad M(u,v) = min(u,v)$$

are copulas and are called $Fre\acute{}chet - Hoefding$ bounds. We refer to M the upper and W the lower $Fre\acute{}chet - Hoeffding$ bounds respectively. Another important copula that we frequently encounter is the product copula, $\Pi(u,v) = uv$.

2.1. Archimedean Copulas

Basic properties of AC are presented below. The more information could be found in Nelsen [11].

Let φ be a continuous, strictly decreasing function from I to $[0,\infty]$ such that $\varphi(1) = 0$. The pseudo-inverse of φ is given by

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{(-1)}(t), & 0 \leq t \leq \varphi(0) \\ 0, & \varphi(0) < t \leq \infty \end{cases}$$

copulas of the form $C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v))$ for every u, v in I, are called AC and the function φ is called a generator of this copulas. If $\varphi[0] = \infty$ we say that φ is a strict generator. In this case, $\varphi^{[-1]} = \varphi^{-1}$ and $C(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v))$ is said a strict Archimedean copula.

3. NEW FAMILY OF ARCHIMEDEAN COPULAS

As we know, any generator φ of an Archimedean copula is convex and strictly decreasing function on [0,1] means it satisfies in the following properties (for any t in I):

$$\varphi(1) = 0, \quad \varphi'(t) < 0, \quad \varphi''(t) < 0. \qquad (3.1)$$

Let $\varphi(t) = csch(t^\theta) - csch(1)$ it can be easily check that $\varphi(t)$ has two continuous derivatives on (0,1) and satisfies in 3.1 on $\theta \in (0,\infty)$.

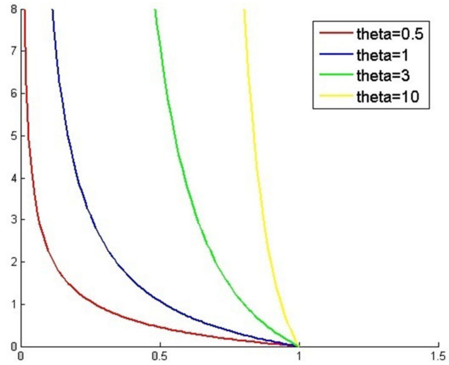

Figure 1. Generator graphs for several values of parameter.

For φ satisfying on the properties 3.1 is enough to guarantee that, it has an inverse φ^{-1} also has two derivatives and is convex and strictly decreasing function on [0,1] (see Figure 1). For convenience, we write $\varphi(0) = \infty$ if $lim_{t\to 0^+} \varphi(t) = \infty$. Let Φ be the class of φ for all $\theta \in (0,\infty)$. Then every member φ of Φ generates a bivariate Archimedean copula as follows

$$C(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v))$$
$$= [acsch(csch(u^\theta) + csch(v^\theta) - csch(1)]^{1/\theta}$$

since $\varphi(0) = \infty$ thus it is a strict generator and $C(u,v)$ is strict Archimedean copula. Random samples generated from the new family are shown in Figure 4. Also Figure 5 demonstrates surfaces and contour diagrams of this new Archimedean family for several values of θ parameter. In generating random samples, surfaces and contour diagrams, Matlab R2008a software, had been used.

Since obtaining a closed form of the Kendall's tau is difficult for the new Archimedean family, hence we used Eviews (6) to obtain the best fitted model for Kendall's tau (see Figure 2). In our process, we selected $\theta = 1, 2, \ldots, 100$ in the new Archimedean family and calculated Kendall's tau for generated random samples in each values of the parameter. Then with trial-and-error, we captured the best fitted model as below

$$\tau_C \cong \frac{\theta}{\theta + 2}$$

for the fitted model, Adjusted R-squared=0.9987, also for assurance, we compared the distribution of actual and fitted data with two-sample Kolmogorov-Smirnov test and obtained, Asymp. Sig. (2-tailed)=0.999, means this test also uphold the fitness of our model. As a last remark in this new Archimedean family $\tau_C \in (0,1)$, namely generated new family is limited to the positive dependency. As copulas are invariant under strictly monotone transformations of the random variables, hence it is possible to generalize the new family to include negative dependency by looking at the copula of $(-U,V)$ where the copula of (U,V) is a member of the family introduced in this manuscript (see Figure 3).

3. CONCLUSION

We defined $\varphi(t) = csch(t^\theta) - csch(1)$ that satisfies on Archimedean copulas generator properties 3.1, thus can generate a new Archimedean family. This new Archimedean family is as below

$$\begin{aligned} C(u,v) &= \varphi^{-1}(\varphi(u)+\varphi(v)) \\ &= [acsch(csch(u^\theta)+csch(v^\theta)-csch(1))]^{1/\theta} \end{aligned}$$

As $\varphi(0) = \infty$ then φ is a strict generator and $C(u,v)$ is a strict Archimedean copula. Also surface, contour graphs and scatterplots of simulated random samples in this new Archimedean family investigated for several values of the family parameter. Since obtaining a closed form of the Kendall's tau is difficult for new family, hence with using Matlab and Eviews softwares, and with trial-and-error we captured the best fitted model as below:

$$\tau_C \cong \frac{\theta}{\theta+2}$$

for the fitted model Adjusted R-squared=0.9987. As a last remark, in this new Archimedean family $\tau_C \in (0,1)$, it means that the generated new family is limited to positive dependency.

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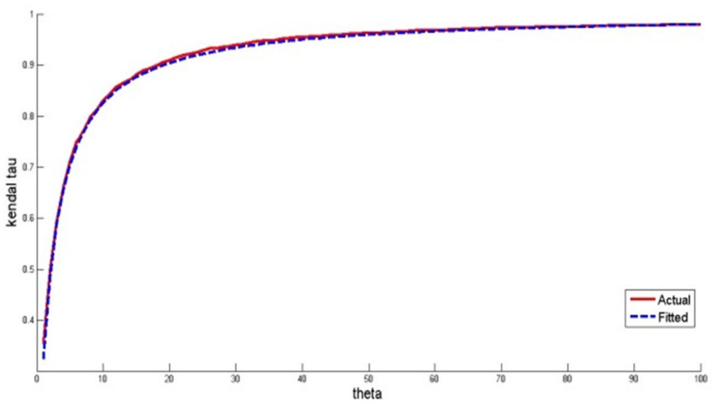

Figure 2. Outline of Kendall's tau for Actual and Fitted data.

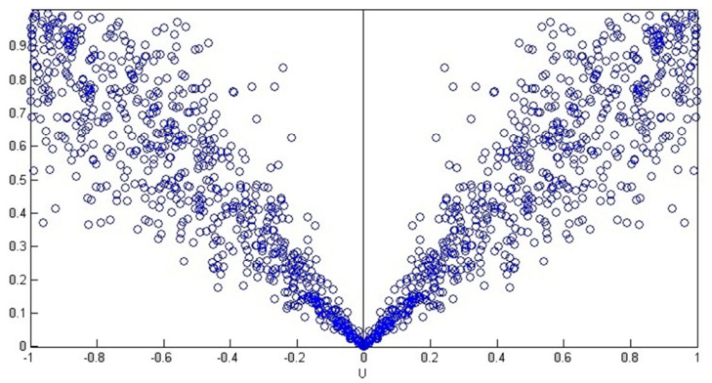

Figure 3. Scatterplots of (U,V) in the new family $\theta = 5$ (right) and $(-U,V)$ (left), n=700.

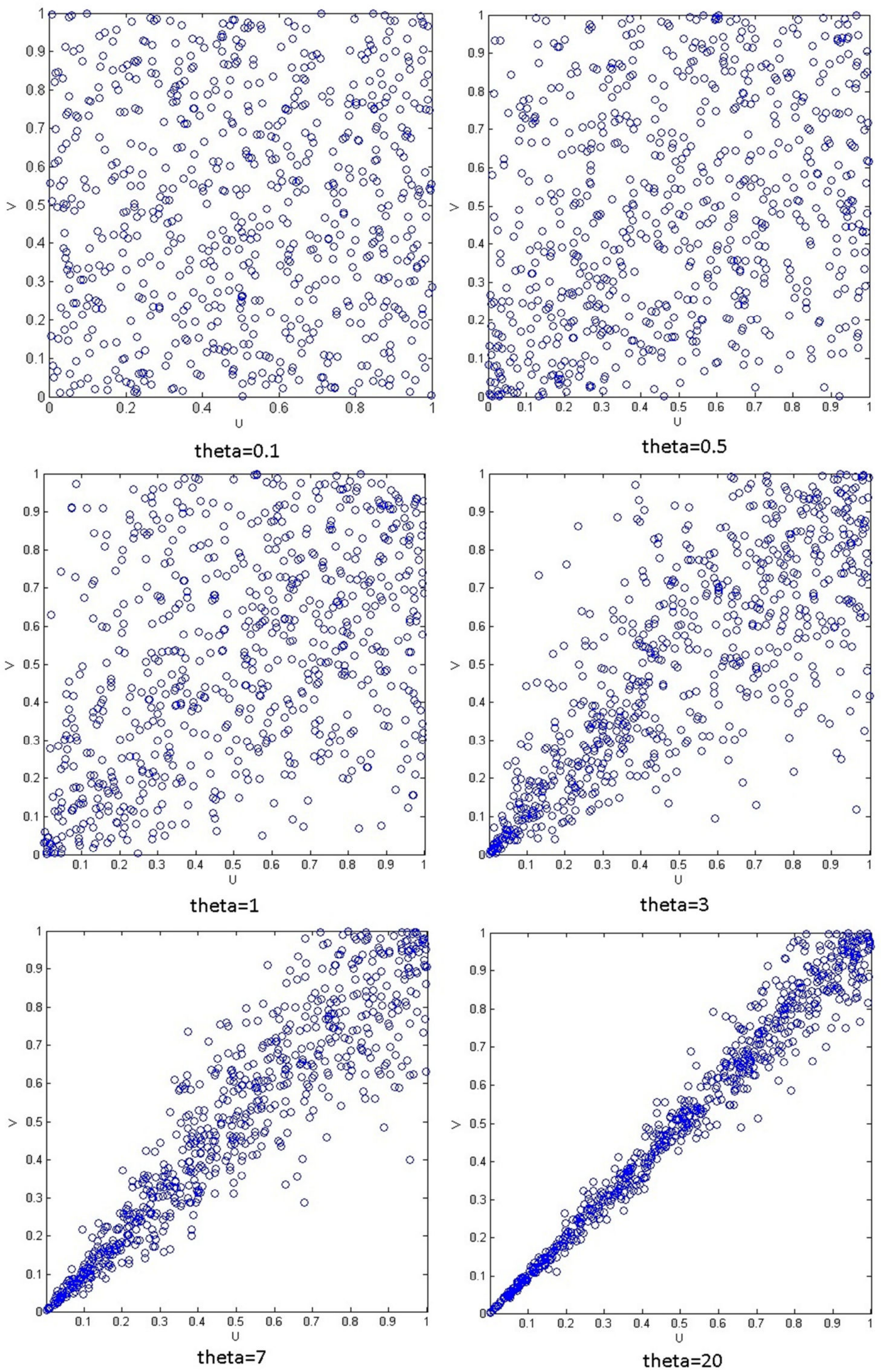

Figure 4. Scatterplots of the new Archimedean family for several values of its parameter, n=700.

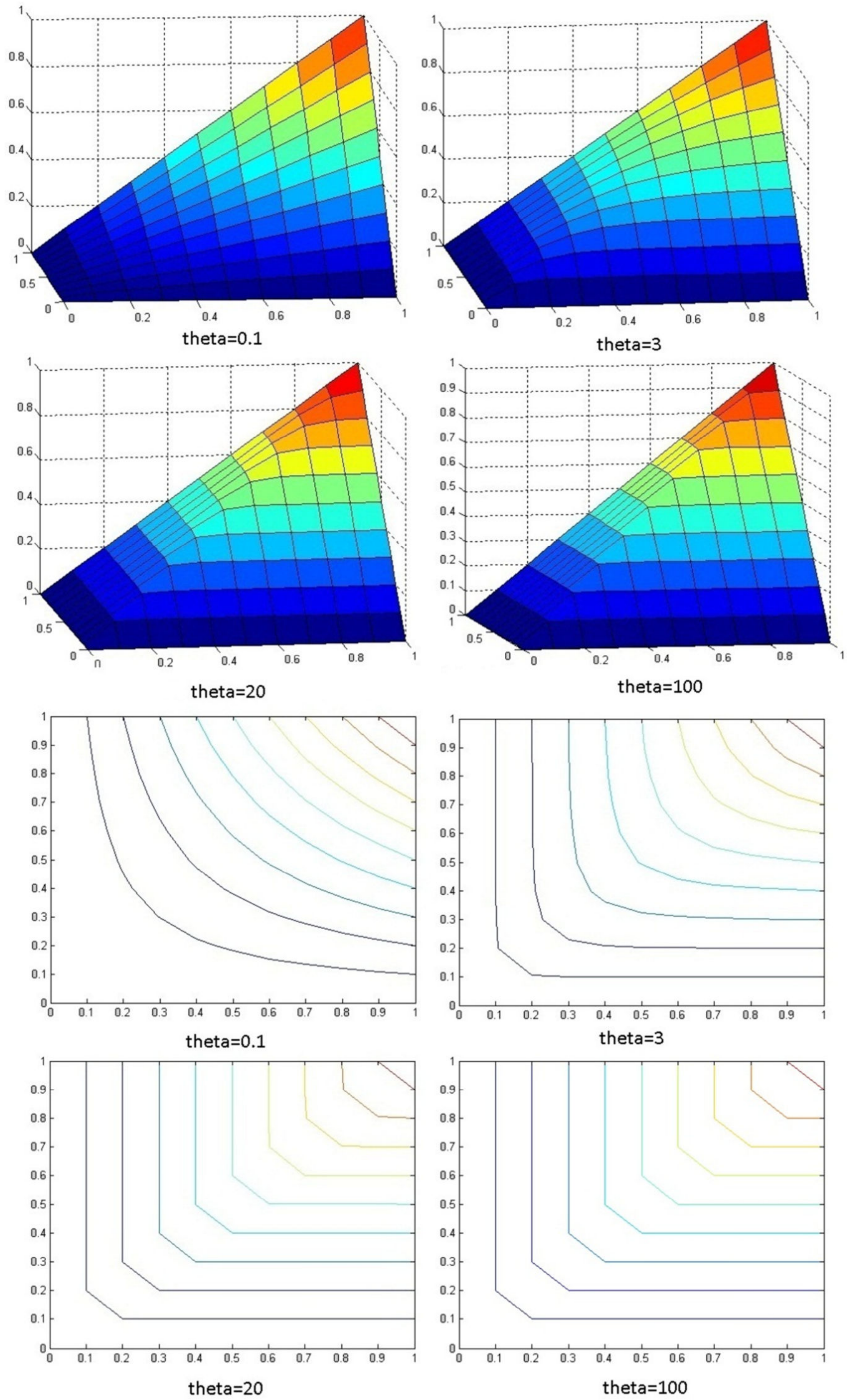

Figure 5. Surfaces and contour diagrams of the new Archimedean family for several values of its parameter.

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